Journal of Statistical Physics, Vol. 86, Nos. 5/6, 1997

A Strong Law of Large Numbers for Iterated Functions of Independent Random Variables

Jan Wehr[1]

Received June 18, 1996

We study sequences of random variables obtained by iterative procedures, which can be thought of as nonlinear generalizations of the arithmetic mean. We prove a strong law of large numbers for a class of such iterations. This gives rise to the concept of generalized expected value of a random variable, for which we prove an analog of the classical Jensen inequality. We give several applications to models arising in mathematical physics and other areas.

KEY WORDS: Law of large numbers; hierarchical models; disordered systems; self-averaging; martingales.

INTRODUCTION

In the last 25 years probability theory has been significantly influenced by ideas from mathematical physics, in particular by the renormalization group theory of K. Wilson (for a discussion of relevant physical ideas see, e.g., refs. 21 and 11. It has been pointed out that the classical theory of addition of independent random variables can be thought of in the renormalization group framework.[9] While implementing the renormalization group ideas at a rigorous mathematical level has turned out to be in general a very difficult task, one class of models—the hierarchical models—is simpler in that it has some renormalization ideas "built in." At the same time, hierarchical models are far from trivial and display rich behavior believed to be representative for more complicated models in the areas of statistical physics, disordered systems, and quantum field theory. This has resulted in a great deal of interest in the mathematical analysis of various types of hierarchical models. In particular, Dyson-type models[7] have been

[1] Department of Mathematics, University of Arizona, Tucson, Arizona 85721.

0022-4715/97/0300-1373$12.50/0

extensively studied; see, e.g., refs. 4 and[17] for some spectacular achievements and ref. 10 for more recent work.

In this paper we study another type of model, equally often encountered in applications (see references below). In these models a sequence of random variables is defined inductively, using a fixed function f of k variables. Given a random variable X at the stage n, the next variable is $f(X_1,..., X_k)$, where $X_1,..., X_k$ are independent copies of X. Hierarchical models of this type are widely used in the theory of disordered systems and in several other applied areas (see references in Section 1). It is thus important to develop a theory studying these models from a rigorous mathematical point of view. This paper goes in this direction by establishing a general convergence theorem which generalizes the classical law of large numbers and at the same time has a clear physical interpretation—that of self-averaging. We illustrate the theorem with various applications to specific models. The related natural (and more complicated) question of central limit theorems for such sequences will be studied in a companion paper.[20]

1. THE MAIN RESULT

Let $\mathscr{X} = (X_1, X_2,...)$ be a sequence of i.i.d. random variables on a probability space $(\Omega, \mathscr{F}, P)$ and let f be a real-valued function of k real variables. We will use f to define another sequence of i.i.d. random variables $\mathscr{Y} = (Y_1, Y_2,...)$ as follows

$$Y_1 = f(X_1,..., X_k) \tag{1.1}$$

$$Y_2 = f(X_{k+1},..., X_{2k}) \tag{1.2}$$

$$\cdots$$

$$Y_l = f(X_{(l-1)k+1},..., Y_{lk}) \tag{1.3}$$

$$\cdots$$

We will denote the resulting sequence $\mathscr{Y}$ by $\mathscr{R}\mathscr{X}$. We thus have a map $\mathscr{R}$ on the space of i.i.d. sequences of random variables. This is similar to block-spin transformations used in the study of spin systems and quantum field theory models. The difference is that in those models the spins are independent and the transformation is linear, whereas here, conversely, independent variables are "blocked" in a nonlinear fashion. Iterating this map with the initial condition $\mathscr{X}^{(0)}$, we obtain a sequence $\mathscr{X}^{(0)}, \mathscr{X}^{(1)},...$ of i.i.d. sequences, where $\mathscr{X}^{(n+1)} = \mathscr{R}\mathscr{X}^{(n)}$ for $n = 0, 1,....$ We will write

$$\mathscr{X}^{(n)} = (X_1^{(n)}, X_2^{(n)},...) \tag{1.4}$$

We are interested in the asymptotics of the distribution of $X_1^{(n)}$ and in the behavior with probability one of the sequence $(X_1^{(0)}, X_1^{(1)},..., X_1^{(n)},...)$ as $n \to \infty$.

In the special case when $X_1^{(0)}$ is integrable with $E(X_1^{(0)}) = \mu$ and $f(x_1,..., x_n) = (x_1 + \cdots + x_k)/k$ the weak law of large numbers implies that

$$X_1^{(n)} \xrightarrow{d} \mu \tag{1.5}$$

and the strong law of large numbers says that

$$P(X_1^{(n)} \to \mu) = 1 \tag{1.6}$$

Our problem can thus be thought of as an analog of this classical situation for iterations of nonlinear averaging operations.

Sequences obtained in the above way are widely studied in statistical physics and related fields, where they come from the so-called hierarchical models. These are models in which the renormalization group approach developed by K. Wilson takes a particularly simple form. For example, models in equilibrium statistical mechanics of disordered systems are discussed in ref. 5. Hierarchical models of random resistor networks have been studied extensively in the literature; see, e.g., refs. 19, 2, 14, and 15. In material science hierarchical sequences of random variables appear as mathematical models of fiber strength and material failure.[12] Yet another example comes from the so-called biased coin problem.[3] In these applications the function f used to define the sequence has some properties of a mean. It is often homogeneous of degree one:

$$f(cx_1,..., cx_n) = cf(x_1,..., x_n) \tag{1.7}$$

and satisfies

$$f(c, c,..., c) = c \tag{1.8}$$

Depending on the particular model, it may also have some convexity, symmetry, or subadditivity properties.

A general theorem about a class of hierarchical sequences of random variables has been proven by Shneiberg.[16] He assumed that f was taking values in the unit interval, was homogeneous of degree one under multiplication by positive numbers, convex separately in each variables and satisfied a normalization condition $f(1,..., 1) = 1$. Under these conditions he proved that the sequence $X_1^{(n)}$ converges to a constant in probability [in fact, also in $L^2(P)$]. His theorem applies, for example, to hierarchical resistor networks with bounded conductivities.

In this paper we study some other (but not disjoint) classes of iterations, for which we prove a strong form of the law of large numbers, i.e., almost sure convergence of the sequence $X_1^{(n)}$ (in the above notation) to a constant.

Theorem 1. Suppose that f is uniformly bounded below and satisfies

$$f(x_1,..., x_k) \leqslant \frac{x_1 + \cdots + x_k}{k} \tag{1.9}$$

for any real $x_1,..., x_k$ in the domain of its definition. If the variables $X_n^{(0)}$ have a finite mean, then the sequence $X_1^{(n)}$ converges almost surely to a constant.

Remark. The proof below has been inspired by the well-known method of proving the law of large numbers using reversed-time martingales.[16] An additional difficulty comes from the fact that f is not assumed to be symmetric. While in most natural applications it has some symmetry properties (see below), which would simplify the proof, we prove the theorem without such additional assumptions.

Proof. We assume that $f \geqslant 0$; the proof in the general case follows the same idea. We swill first prove that the sequence

$$Z_n = \frac{X_1^{(n)} + \cdots + X_k^{(n)}}{k} \tag{1.10}$$

converges almost surely to a constant. Let us define $\mathscr{F}_n$ to be the σ-algebra generated by the variables $X_1^{(n)} + \cdots + X_k^{(n)}, X_{k^{n+1}+1}^{(0)}, X_{k^{n+1}+2}^{(0)},...$. Clearly, the $\mathscr{F}_n$ form a decreasing sequence of σ-algebras and, for each n, Z_n is $\mathscr{F}_n$-measurable. Furthermore, by the obvious symmetry,

$$\begin{aligned} E(Z_n \mid \mathscr{F}_{n+1}) &= \frac{1}{k} E(X_1^{(n)} + \cdots + X_k^{(n)} \mid X_1^{(n+1)} + \cdots X_k^{(n+1)}) \\ &= \frac{1}{k^2} E(X_1^{(n)} + \cdots X_{k^2}^{(n)} \mid X_1^{(n+1)} + \cdots X_k^{(n+1)}) \end{aligned} \tag{1.11}$$

By (1.9), the last expression is bounded below by

$$\frac{1}{k} E(X_1^{(n+1)} + \cdots + X_k^{(n+1)} \mid X_1^{(n+1)} + \cdots + X_k^{(n+1)}) = Z_{n+1} \tag{1.12}$$

We have thus shown that the sequence (Z_n) is a reversed-time submartingale relative to the family $(\mathscr{F}_n)$ of σ-algebras (see ref. 13 for the definition and fundamental theorems about reversed-time submartingales). Since $Z_n \geqslant 0$, the convergence theorem for nonnegative reversed-time submartingales[13] implies that Z_n converges almost surely to a random variable Z. We will next show that Z is constant with probability one. The classical Hewitt–Savage zero–one law says that a measurable function of independent and identically distributed random variables $X_1^{(0)}, X_2^{(0)},\ldots,$ invariant under any finite permutation of the indices, is almost surely equal to a constant. While Z does not satisfy this symmetry condition, it is clearly invariant under some special permutations of indices, in particular, for each n it does not change its value under the permutation which transposes 1 with k^n+1, 2 with $k^n+2,\ldots, k^n$ with $2k^n$, and leaves all the other indices unchanged. The standard proof of the Hewitt–Savage law (e.g., in ref. 6) shows that this weaker symmetry is enough to prove that Z is almost surely constant (in other words, the assumptions of the Hewitt–Savage theorem can be weakened to apply to the present situation without changing the proof). We have thus shown that Z_n converges almost surely to a constant. We now want to show that this is also true for $X_1^{(n)}$. It is clearly enough to show this in the case when the limit of Z_n is zero, which from now on we assume for notational convenience. Since Z_n converges to zero in probability and since, by independence of $X_1^{(n)},\ldots, X_k^{(n)}$,

$$P(Z_n > \varepsilon) \geqslant P(X_1^{(n)} > \varepsilon)^k \tag{1.13}$$

it follows that $X_1^{(n)}$ converges to zero in probability (and, consequently, the same is true about $X_l^{(n)}$ for all $l \leqslant k$). Let us first assume that all $X_l^{(n)}$ are bounded in absolute value. The general case is proven below using the same idea, but the proof is less transparent, due to a truncation. Note that the variables $X_2^{(n)},\ldots, X_k^{(n)}$ are independent from the σ-algebra $\mathscr{G}_n$ generated by $X_1^{(0)},\ldots, X_1^{(n)}$. If we denote by $\mathscr{G}_\infty$ the smallest σ-algebra generated by all $\mathscr{G}_n$, the dominated convergence theorem for conditional expectations[6] implies that with probability one

$$X_1^{(n)} = E[kZ_n \mid \mathscr{G}_n] - (k-1)\, E[X_1^{(n)} \to E[0 \mid \mathscr{G}_\infty] - (k-1)\, c = c \tag{1.14}$$

since by the (unconditional) bounded convergence theorem, almost sure convergence of $Z_n \to 0$ implies that $E[X_1^{(n)}] \to 0$. This proves the bounded case. The general case is handled in a similar way, except that we need to introduce the truncated variables: for a random variable Y, Y^* will denote -1, Y, or 1 according to whether $Y < -1$, $-1 \leqslant Y \leqslant 1$, or $Y > 1$, respectively. Let $V_n = X_2^{(n)} + \cdots + X_k^{(n)}$. Since by assumption, $X_1^{(n)} + V_n \to 0$, it follows that also $(X_1^{(n)})^* + V_n^* \to 0$. Introducing the σ-algebras $\mathscr{G}_n$ as above, we obtain $(X_1^{(n)})^* + \mu_n \to 0$ almost surely, where $\mu_n = E(V_n^*)$ is a sequence of constants. It follows that $(X_1^{(n)})^* + \mu_n \to 0$ in probability. Since $X_1^{(n)} \to 0$ in probability, we also have $(X_1^{(n)})^* \to 0$ in probability and it follows that $\mu_n \to 0$, which implies that with probability one $(X_1^{(n)})^*$ goes to zero and the same follows for $X_1^{(n)}$, which completes the proof.

2. APPLICATIONS AND EXTENSIONS

2.1. Random Resistor Networks

Theorem 1 applies to several hierarchical networks of random resistors. As an examples we consider the so-called diamond network, which is an iteration of the parallel connection of two pairs of sequentially connected resistors. The function f corresponding to this iteration is

$$f(x_1, x_2, x_3, x_4) = \frac{1}{1/x_1 + 1/x_2} + \frac{1}{1/x_3 + 1/x_4} \tag{2.1}$$

where the x_i are nonegative (and $1/0 = +\infty$; $1/(+\infty) = 0$). The x_i represent conductivities of the random resistor and $f(x_1, x_2, x_3, x_4)$ is the effective conductivity of the system of four resistors arranged in a "diamond." It is easy to see, using the inequality between harmonic and arithmetic means, that

$$f(x_1, x_2, x_3, x_4) \leqslant \frac{x_1 + x_2 + x_3 + x_4}{4} \tag{2.2}$$

We can thus apply Theorem 1 and conclude that the sequence $X_1^{(n)}$ of variables obtained by iterations of f converges almost surely to a constant. This constant has the interpretation of the effective conductivity of the hierarchical diamond lattice in the infinite-volume limit. The above iteration has been extensively studied, numerically, analytically, and by rigorous computer-assisted methods.[19, 2, 14, 15] The result of ref. 16 mentioned above applies here, giving convergence in L^2. Moreover, it follows from ref. 16 that the limit is strictly positive if and only if $P[X_1^{(0)} > 0] > p_c$, where

$$p_c = \frac{\sqrt{5} - 1}{2} \tag{2.3}$$

Physical heuristics based on renormalization group theory suggests that for $p > p_c$, suitably normalized random variables $X_1^{(n)} - E[X_1^{(n)}]$ should converge to a normal distribution, whereas at $p = p_c$ a non-Gaussian behavior is likely. The first part of this conjecture is addressed (for a general class of iterations) in ref. 20. A non-Gaussian fixed point of the corresponding renormalization group map is constructed in ref. 15 by a (rigorous) computer-assisted proof.

The above convergence proof applies to many other hierarchical resistor networks, but not to the general case studied in ref. 16, since not all functions f coming from such networks satisfy the condition (1.9). On the other hand, whenever Theorem 1 applies, it gives a stronger result (almost sure convergence).

2.2. Durability of Hierarchical Fibers

In this model one studies fibers which can sustain an external force up to a random breaking point (see ref. 12 and references therein). Given two such fibers with breaking points X_1 and x_2, the composite fiber has the breaking point defined by the following rule: an applied force is equally distributed to the two halves. If one fiber breaks, the other fiber inherits the force held by the other, i.e., it now has to sustain the entire load (and may also break if the load exceeds its breaking point). It is easy to see that the new breaking point is given by

$$2(x_1 \wedge x_2) \vee x_1 \vee x_2 \tag{2.4}$$

where $x \wedge y$ denotes the minimum of x and y and $x \vee y$ the maximum of x and y. If $x_1 = x_2 = a$, the new breaking point is clearly $2a$ and it is thus natural to divide the above expression by 2. We are thus led to an iteration scheme with

$$f(x_1, x_2) = (x_1 \wedge x_2) \vee \frac{x_1}{2} \vee \frac{x_2}{2} \tag{2.5}$$

It is easy to see that

$$f(x_1, x_2) \leqslant \frac{x_1 + x_2}{2} \tag{2.6}$$

so if the variables $X_n^{(0)}$ are arbitrary nonnegative i.i.d. random variables with $E(X_n^{(0)}) < \infty$, Theorem 1 guarantees convergence of the sequence $X_1^{(n)}$ to a constant. Note that f is neither convex nor concave as a function of any of the variables, so the results of ref. 16 do not apply in this case.

2.3. The Biased Coin Problem

A sequence of random variables obtained by iterating the function $f(x_1, x_2) = (1 - \varepsilon)(x_1 \wedge x_2) + \varepsilon(x_1 \vee x_2)$ was considered in ref. 3 in relation to the so-called biased coin problem. It is easy to see that for $\varepsilon \leqslant 1/2$ we again have

$$f(x_1, x_2) \leqslant \frac{x_1 + x_2}{2} \tag{2.7}$$

so that Theorem 1 applies again with any integrable variables $X_n^{(0)}$. This strengthens a result obtained by different means in ref. 3.

2.4. Relation to the Subadditive Ergodic Theorem and Comments on the Superadditive Case

In some sense Theorem 1 is similar to Kingman's subadditive ergodic theorem.[6] In fact, the (multiparameter) subadditive ergodic theorem[1] can be used to prove almost sure convergence to an effective parameter in lattice models of statistical mechanics, conductivity, etc.,[8, 18] in analogy with the application in Section 2.1. Due to lack of enough translational invariance in the assumptions of Theorem 1, we do not see how to derive it from Kingman's theorem; another approach, based on martingale theory, seemed more appropriate.

There is, of course, a theorem analogous to Theorem 1 for the case when

$$f(x_1,..., x_k) \geqslant \frac{x_1 + \cdots + x_k}{k} \tag{2.8}$$

In view of the above analogy with Kingman's theorem, it is natural to call an iteration like this "superadditive." If we want to use Theorem 1 with no extra work, however, we need to assume in the superadditive case that f is bounded above, a condition that is not always satisfied in interesting applications. For example, if in the biased coin problem $\varepsilon \geqslant 1/2$, we have

$$f(x_1, x_2) \geqslant \frac{x_1 + x_2}{2} \tag{2.9}$$

and consequently $X_1^{(n)}$ is a reversed-time supermartingale. Even though the function f is clearly not bounded above, if the variables $X_i^{(0)}$ are bounded above, so is the resulting supermartingale and this implies almost sure convergence.[13]

2.5. Jensen-Type Inequalities

Given a random variable X and a function f such that Theorem 1 applies to iterations of f, starting from a sequence $X_i^{(0)}$ in which all variables have the distribution of X, it is natural to interpret the limit of the sequence $X_1^{(n)}$ as a nonlinear average of X. In particular, if $f(x_1,\ldots, x_k) = (x_1 + \cdots + x_k)/k$, the limit (for an integrable initial distribution) is $E[X]$, the mean of the variable X (by the law of large numbers). In the general case we will denote the limit by $f^*(X)$. In models of disordered systems $f^*(X)$ has the interpretation of an effective parameter (e.g., conductivity) of the model, arising in the infinite-volume limit as a result of a complicated averaging process applied to constituents of the system. It is natural to expect that $f^*(X)$ shares some properties of the expected value [note, however, that $f^*(X)$ can vanish for a nonnegative variables, even when the variable is not identically zero; this happens, for example, in the diamond network case when $p < p_c$].

Definition. A function g is called f-convex if for any $x_1,\ldots, x_k$,

$$g(f(x_1,\ldots, x_k)) < f(g(x_1),\ldots, g(x_k)) \tag{2.10}$$

Of course, in the case when f is the arithmetic mean, f-convexity is the usual convexity.

Proposition. Assume that Theorem 1 applies to f and X in the above sense. Let g be f-convex, bounded below, continuous, increasing, and $E[g(X)] < \infty$. Then

$$g(f^*(X)) \leqslant f^*(g(X)) \tag{2.11}$$

Remark. This is a generalization of the usual Jensen inequality, which we obtain in the case when f^* is the usual expectation.

Proof. Let $Y_i^{(n)}$ denote the family obtained by iterations of the function f with $Y_i^{(0)} = g(X_i^{(0)})$. We will prove by induction of n that

$$g(X_1^{(n)}) \leqslant Y_1^{(n)} \tag{2.12}$$

For $n = 0$ this follows from the definition of $Y_1^{(0)}$. If the statement is true for n, then using f-convexity and monotonicity of g, we obtain

$$\begin{aligned} g(X_1^{(n+1)}) &= g(f(X_1^{(n)},\ldots, X_k^{(n)})) \\ &\leqslant f(g(X_1^{(n)}),\ldots, g(X_k^{(n)})) \\ &\leqslant f(Y_1^{(n)},\ldots, Y_k^{(n)}) = Y_1^{(n+1)} \end{aligned} \tag{2.13}$$

which proves (2.12). Taking the limit and using continuity of g, we get (2.11), as claimed.

Example. Let f be the diamond average considered above. Then a C^2-function g is f-convex if and only if: (i) g is convex (in the usual sense), and (ii) the function $\phi(x) = 1/g(1/x)$ is concave.

Proof. Suppose first that (i) and (ii) are satisfied. Then, for any $x_1, x_2, x_3, x_4 \geqslant 0$, concavity of ϕ easily implies

$$g\left(\frac{2}{1/x_1 + 1/x_2}\right) \leqslant \frac{2}{1/g(x_1) + 1/g(x_2)} \tag{2.14}$$

and

$$g\left(\frac{2}{1/x_3 + 1/x_4}\right) \leqslant \frac{2}{1/g(x_3) + 1/g(x_4)} \tag{2.15}$$

Hence, convexity of g gives

$$g\left(\frac{1}{1/x_1 + 1/x_2} + \frac{1}{1/x3 + 1/x_4}\right) = g\left(\frac{1}{2}\left[\frac{2}{1/x_1 + 1/x_2} + \frac{2}{1/x_3 + 1/x_4}\right]\right) \tag{2.16}$$

$$\leqslant \frac{1}{2}\left[g\left(\frac{2}{1/x_1 + 1/x_2}\right) + g\left(\frac{2}{1/x_3 + 1/x_4}\right)\right]$$

$$\leqslant \frac{1}{2}\left[\frac{2}{1/g(x_1) + 1/g(x_2)} + \frac{2}{1/g(x_3) + 1/g(x_4)}\right] \tag{2.17}$$

which is f-convexity of g.

Conversely, the f-convexity of g

$$g\left(\frac{1}{1/x_1 + 1/x_2} + \frac{1}{1/x_3 + 1/x_4}\right) \leqslant \frac{1}{1/g(x_1) + 1/g(x_2)} + \frac{1}{1/g(x_3) + 1/g(x_4)} \tag{2.18}$$

with $x_1 = x_2$ and $x_3 = x_4$, implies (i), and with $x_1 = x_3$ and $x_2 = x_4$ it implies (ii), which ends the proof.

Using the criterion above, we can show an example of a nontrivial f-concave function: $g(x) = \arctan(x)$, $x \geqslant 0$.

3. SUMMARY

We have proven a law of large numbers for sequences obtained by iterations, including several models used in applied sciences. This allows us to introduce a concept of nonlinear averages, for which we prove an analog of Jensen's inequality. A natural question of limit theorems for the suitably scaled sequence $X_1^{(n)}$ will be addressed in a companion paper.[20]

ACKNOWLEDGMENTS

I am grateful to. H. Spohn, who invited me to the University of Munich and showed me his work on the diamond lattice, which was the original motivation for the present paper. Thanks are also due to J. Bellissard, R. Blumenfeld, R. Boppana, E. Carlen, G. Grimmett, F. Guerra, and G. Swindle for helpful comments at various stages of this work, and to A. Schenkel and P. Wittwer for the privilege of their collaboration on a related project. The hospitality of E. Lieb in Princeton is gratefully acknowledged, where the work was started and partially supported by NSF grant PHY 90 19433 A01.

REFERENCES

1. M. Akcoglu and J. Krengel, Ergodic theorems for superadditive processes, *J. Reine Angew. Math.* **323**:53 (1981).
2. R. Blumenfeld, Probability densities of homogeneous functions: Explicit approximations and applications to percolating networks, *J. Phys. A* **21**:815 (1988).
3. R. Boppana and B. Narayan, On the biased coin problem, in *Proceedings of the 25th ACM Symposium on Theory of Computing* (ACM Press, New York, 1993).
4. P. Collet and J.-P. Eckmann, *A Renormalization Group Analysis of the Hierarchical Model in Statistical Mechanics* (Springer, New York, 1978).
5. B. Derrida, Pure and random models of statistical mechanics on hierarchical lattices, in *Critical Phenomena, Random Systems and Gauge Theories*, K. Osterwalder and R. Stora, eds. (North-Holland, Amsterdam, 1986).
6. R. Durrett, *Probability. Theory and Examples*, 2nd ed. (Duxbury Press, Belmont, California, Wadsworth 1996).
7. F. Dyson, Nonexistence of spontaneous magnetization in a one-dimensional Ising Ferromagnet, *Commun. Math. Phys.* **12**:212 (1969).
8. J. M. Hammersley, Mesoadditive processes and the specific conductivity of lattices, *J. Appl. Prob.* **25A**:347 (1988).
9. G. Jona-Lasinio, The renormalization group: A probabilistic view, *Nuovo Cimento* **26B**:99 (1975).
10. H. Koch and P. Wittwer, A nontrivial renormalization group fixed point for the Dyson–Baker hierarchical model, *Commun. Math. Phys.* **164**:627 (1994).
11. S.-K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, 1976).
12. W. I. Newman *et al.*, An exact renormalization model for earthquakes and material failure. Statics and dynamics, *Physica D* **77**:200 (1994).
13. L. Rogers and D. Williams, *Diffusions, Martingales and Markov Processes* (Wiley, New York, 1994).
14. T. Schlösser and H. Spohn, Sample-to-sample fluctuations in the conductivity of a disordered medium, *J. Stat. Phys.* **69**:959 (1992).

15. A. Schenkel, J. Wehr, and P. Wittwer, A Non-Gaussian renormalization group fixed point in a hierarchical model of random resistors, in preparation.
16. I. Shneiberg, Hierarchical sequences of random variables, *Theory Prob. Appl.* **31**:137 (1986).
17. Y. Sinai, *The Theory of Phase Transitions: Rigorous Results* (Pergamon Press, Oxford, 1981).
18. T. Spencer, The Schrödinger equation with a random potential, in *Critical Phenomena, Random Systems and Gauge Theories*, K. Osterwalder and R. Stora, eds. (North-Holland, Amsterdam, 1986).
19. R. B. Stinchcombe and B. P. Watson, Renormalization group approach for percolation conductivity, *J. Phys. C* **9**:3221 (1976).
20. J. Wehr, On convergence to a normal distribution of sequences obtained by iterations of nonlinear functions, in preparation.
21. K. Wilson and J. Kogut, The renormalization group and ε-expansion, *Phys. Rep.* **12C**:75 (1975).